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On generalized Bäcklund transformations for equations describing pseudo-spherical surfaces

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Abstract

It is shown that each one-parameter subgroup of $SL(2, \mathbf{R})$ gives rise to a local correspondence theorem between suitably generic solutions of arbitrary scalar equations describing pseudo-spherical surfaces. Thus, if appropriate genericity conditions are satisfied, there exist local transformations between *any two* solutions of scalar equations arising as integrability conditions of $sl(2, \mathbf{R})$-valued linear problems.

A complete characterization of evolution equations $u_t = K(x, t, u, u_x, \dots, u_{x^k})$ which are of strictly pseudo-spherical type is also provided.

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1. Introduction

A scalar differential equation $\Xi(x, t, u, \dots) = 0$ is of *pseudo-spherical type* (or, it describes pseudo-spherical surfaces) if there exist functions f_{ij}, $i = 1, 2, 3$, $j = 1, 2$, depending on x, t, u and a finite number of derivatives of u, such that the one-forms $\omega^i = f_{i1}\,\mathrm{d}x + f_{i2}\,\mathrm{d}t$ satisfy the structure equations

$$\mathrm{d}\omega^1 = \omega^3 \wedge \omega^2, \qquad \mathrm{d}\omega^2 = \omega^1 \wedge \omega^3, \qquad \mathrm{d}\omega^3 = \omega^1 \wedge \omega^2, \tag{1}$$

whenever $u(x, t)$ is a solution of $\Xi = 0$. If $u : M \subseteq \mathbf{R}^2 \to \mathbf{R}$ satisfies $(\omega^1 \wedge \omega^2)(u(x, t)) \neq 0$—what will be called “III-genericity” in Section 2—the structure equations (1) imply that

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$\{\omega^1(u(x,t)), \omega^2(u(x,t))\}$ is a moving coframe on M with connection one-form $\omega^3(u(x,t))$, and that the Gaussian curvature of the metric $\omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2$ on M is -1.

In 1995, Kamran and Tenenblat [13] proved a most interesting correspondence theorem for scalar equations describing pseudo-spherical surfaces: motivated by the classical fact that two surfaces of constant Gaussian curvature equal to -1 are locally indistinguishable, they showed that given two such equations, and III-generic solutions $u(x,t)$ and $\hat{u}(\hat{x},\hat{t})$ of them, one can relate $u(x,t)$ and $\hat{u}(\hat{x},\hat{t})$ by integrating first-order systems of equations. They then obtained a *formula* for $\hat{u}(\hat{x},\hat{t})$ in terms of the function $u(x,t)$ and some "pseudo-potentials" determined solely by $u(x,t)$.

This transformation result goes well beyond the classical Bäcklund theorem [8]: it requires an explicit change of independent variables, while Bäcklund's result involves only a change in the dependent variable; it is not, apparently, tied up to symmetry considerations (see [1,4,8] for discussions on symmetries and Bäcklund correspondences); finally, it is not restricted to transforming solutions of a same equation, or even of equations of the same order.

The main goal of this work is to present some new correspondence results à la Kamran–Tenenblat: they are reported in Section 3. The key observation which allows one to extend the results of [13] is that the structure equations (1) can be also understood in terms of pseudo-Riemannian geometry: if, for instance, $u(x,t)$ is a solution of $\Xi = 0$ such that $(\omega^2 \wedge \omega^3)(u(x,t)) \neq 0$, then $\omega^2 \otimes \omega^2 - \omega^3 \otimes \omega^3$ determines a pseudo-Riemannian metric of Gaussian curvature -1 on the domain of $u(x,t)$. Since two pseudo-Riemannian surfaces of constant Gaussian curvature -1 are locally isometric, one expects a correspondence theorem as in the Riemannian case.

Additional motivation for this paper—and for the consideration of pseudo-Riemannian manifolds in the theory of equations of pseudo-spherical type—comes from the relation between equations in this class and linear problems: as will be seen in Section 2, an equation $\Xi = 0$ of pseudo-spherical type is the integrability condition of an $sl(2,\mathbf{R})$-valued linear problem $v_x = Xv$, $v_t = Tv$. It follows that [5,7,9] if $u(x,t) : M \subseteq \mathbf{R}^2 \to \mathbf{R}$ is a solution of $\Xi = 0$, the matrix valued one-form

$$\Omega(u(x,t)) = X(u(x,t))\,\mathrm{d}x + T(u(x,t))\,\mathrm{d}t \tag{2}$$

is an $sl(2,\mathbf{R})$ connection one-form on the trivial bundle $M \times SL(2,\mathbf{R})$ whose curvature $\Theta = \mathrm{d}\Omega(u(x,t)) - \Omega(u(x,t)) \wedge \Omega(u(x,t))$ vanishes identically. Conversely, if the connection form (2) satisfies $\Theta = 0$ whenever $u(x,t) : M \subseteq \mathbf{R}^2 \to \mathbf{R}$ is a solution of $\Xi = 0$, the structure of a Riemannian surface of Gaussian curvature -1 on M arises as a consequence of "splitting" the one-form $\Omega(u(x,t))$, and identifying a part of this splitting with a moving coframe on M. (Chamseddine and Wyler [3], for instance, use this process in their gauge formulation of dilaton gravity in two dimensions). A *different* splitting of $\Omega(u(x,t))$ will yield, instead, a *pseudo-Riemannian* structure on M.

Now, from this point of view, Kamran and Tenenblat's result [13] holds because the Poincaré metric on the upper half plane determines a *standard* "undressed" [9] $sl(2,\mathbf{R})$ flat connection one-form on (an open subset of) $\mathbf{R}^2 \times SL(2,\mathbf{R})$, Ω_0 say, *and* for any equation $\Xi = 0$ of pseudo-spherical type one can find a (local) diffeomorphism $\Psi : M \to \mathbf{R}^2$ and an $SO(2)$-valued gauge transformation between the connection $\Omega(u(x,t))$ defined in (2) and $\Psi^*\Omega_0$ (see [13] and Section 3). It is then very natural to expect that consideration of pseudo-Riemannian metrics on (open subsets of) $\mathbf{R}^2$, their undressed connection forms, and $SO(1,1)$-valued gauge transformations, will give rise to *different* generalized Bäcklund correspondences.

The second goal of this paper is to contribute to the classification of equations of pseudo-spherical type, a question of intrinsic geometrical interest. A complete description of evolution equations $u_t = K(x, t, u, u_x, \dots, u_{x^k})$ which are "strictly pseudo-spherical" is presented in Section 4, where strictly pseudo-spherical equations are also rigorously introduced. The theorems in this section generalize the classification results known up to now [4,13,19]. It has been decided to report on this material here because gauge transformations also appear in this context. This time, they are used to simplify the analysis made in the articles mentioned above.

Standard notions of formal differential geometry [14,16] will be used throughout. Thus, a trivial bundle E given locally by $(x, t, u) \mapsto (x, t)$ will be fixed once and for all, and a scalar differential equation $\Xi = 0$ in two independent variables x, t, will be often identified with a subbundle S^∞ of the infinite jet bundle $J^\infty E$ of E (see [14]). The following facts [14,16] will be also needed below:

(a) Local solutions of $\Xi = 0$ correspond to local holonomic sections $j^\infty(s)$ of S^∞, in which $s : (x, t) \mapsto (x, t, u(x, t))$ is a local section of E.
(b) The horizontal exterior derivative operator d_H acts on one-forms $\omega = A\,\mathrm{d}x + B\,\mathrm{d}t$ on $J^\infty E$ (resp. S^∞) $d_H\omega$ by means of $d_H f = \mathrm{D}_x f\,\mathrm{d}x + \mathrm{D}_t f\,\mathrm{d}t$ and $d_H(\mathrm{d}x) = d_H(\mathrm{d}t) = 0$, in which $\mathrm{D}_x, \mathrm{D}_t$ are the total derivatives operators on $J^\infty E$ (resp. S^∞).
(c) The operator d_H satisfies the identity
$$d((j^\infty(s))^*\omega) = (j^\infty(s))^*(d_H\omega) \tag{3}$$
for every local section $j^\infty(s)$ of $J^\infty E$ (resp. holonomic section $j^\infty(s)$ of S^∞).

2. Equations describing pseudo-spherical surfaces

The following structure is the point of departure for this paper.

Definition 1. A differential equation $\Xi(x, t, u, \dots, u_{x^m t^n}) = 0$ describes pseudo-spherical surfaces (or, it is of pseudo-spherical type) if there exist smooth functions f_{ij} $(i = 1, 2, 3;\ j = 1, 2)$ on $J^\infty E$ such that the pull-back of the one-forms $\omega^i = f_{i1}\,\mathrm{d}x + f_{i2}\,\mathrm{d}t$ by local solutions $u(x, t)$ of $\Xi = 0$, $\bar\omega^i$ say, satisfy the structure equations
$$\mathrm{d}\bar\omega^1 = \bar\omega^3 \wedge \bar\omega^2, \qquad \mathrm{d}\bar\omega^2 = \bar\omega^1 \wedge \bar\omega^3, \qquad \mathrm{d}\bar\omega^3 = \bar\omega^1 \wedge \bar\omega^2. \tag{4}$$

The trivial case of all functions f_{ij} depending only on the independent variables x, t is excluded from the considerations below. Also, the expression "PSS equation" will be sometimes utilized as an abbreviation of the phrase "equation describing pseudo-spherical surfaces".

Definition 1 was introduced in 1986 by Chern and Tenenblat [4], motivated by Sasaki's [23] observation that equations integrable via the Ablowitz, Kaup, Newell, and Segur (AKNS) scattering/inverse scattering method describe pseudo-spherical surfaces whenever their associated linear problems are real. One can study classical Bäcklund transformations, symmetries, and conservation laws of equations of pseudo-spherical type by geometrical means ([1,4,19–21,24] and references therein), and one can also effectively classify them, as it will be shown in Section 4 (see also [4,13,19,22,24] and references therein).

Example 1. A classical example of PSS equation—besides the sine-Gordon equation [8,24]—is the ubiquitous KdV equation $u_t = u_{xxx} + uu_x$: it describes pseudo-spherical surfaces with associated one-forms

$$\omega^1 = (1-u)\,\mathrm{d}x + (-u_{xx} + \eta u_x - \eta^2 u - \tfrac{1}{3}u^2 + \eta^2 - \tfrac{2}{9}u + \tfrac{5}{9})\,\mathrm{d}t, \tag{5}$$

$$\omega^2 = \eta\,\mathrm{d}x + (\eta^3 + \tfrac{1}{3}\eta u - \tfrac{1}{3}u_x + \tfrac{5}{9}\eta)\,\mathrm{d}t, \tag{6}$$

$$\omega^3 = (\tfrac{2}{3} - u)\,\mathrm{d}x + (-u_{xx} + \eta u_x - \eta^2 u - \tfrac{1}{3}u^2 + \tfrac{2}{3}\eta^2 - \tfrac{1}{3}u + \tfrac{10}{27})\,\mathrm{d}t, \tag{7}$$

in which η is an arbitrary real parameter.

The interpretation of Definition 1 in terms of differential geometry of surfaces is based on the following genericity notions.

Definition 2. Let $\Xi = 0$ be an equation describing pseudo-spherical surfaces with associated one-forms ω^i, $i = 1, 2, 3$. A solution $j^\infty(s)$ of $\Xi = 0$ will be called I-generic if $(j^\infty(s))^*(\omega^3 \wedge \omega^2) \neq 0$; II-generic if $(j^\infty(s))^*(\omega^1 \wedge \omega^3) \neq 0$; and III-generic if $(j^\infty(s))^*(\omega^1 \wedge \omega^2) \neq 0$.

For instance, $u(x,t) = x + t$ is a I- and III-generic solution of the PSS equation $u_t = u_{xx} + u_x$ with associated one-forms $\omega^1 = u\,\mathrm{d}x + u_x\,\mathrm{d}t$, $\omega^2 = \mathrm{d}x$, and $\omega^3 = u\,\mathrm{d}x + u_x\,\mathrm{d}t$, but it is not II-generic. On the other hand, it is II-generic if one considers, instead of the forms ω^i, the associated one-forms $\hat{\omega}^1 = \omega^2$, $\hat{\omega}^2 = -\omega^1$, and $\hat{\omega}^3 = \omega^3$.

Proposition 1. *Let $\Xi = 0$ be a PSS equation with associated one-forms ω^i, $i = 1, 2, 3$, and let $s : (x,t) \mapsto (x, t, u(x,t))$ be a local section of E. Then,*

(a) *If $j^\infty(s)$ is a I-generic solution of $\Xi = 0$, the one-forms $\sigma_1 = j^\infty(s)^*\omega^2$ and $\sigma_2 = j^\infty(s)^*\omega^3$ determine a Lorentzian metric of constant Gaussian curvature $K = -1$ on the domain S of $u(x,t)$, with connection one-form given by $\sigma_{12} = j^\infty(s)^*\omega^1$.*
(b) *If $j^\infty(s)$ is a II-generic solution of $\Xi = 0$, the one-forms $\sigma_1 = j^\infty(s)^*\omega^1$ and $\sigma_2 = -j^\infty(s)^*\omega^3$ determine a Lorentzian metric of constant Gaussian curvature $K = -1$ on the domain S of $u(x,t)$, with connection one-form given by $\sigma_{12} = j^\infty(s)^*\omega^2$.*
(c) *If $j^\infty(s)$ is a III-generic solution of $\Xi = 0$, the one-forms $\sigma_1 = j^\infty(s)^*\omega^1$ and $\sigma_2 = j^\infty(s)^*\omega^2$ determine a Riemannian metric of constant Gaussian curvature $K = -1$ on the domain S of $u(x,t)$, with connection one-form given by $\sigma_{12} = j^\infty(s)^*\omega^3$.*

Proof. The structure equations of a surface S with metric $\mathrm{d}s^2 = \epsilon_1(\sigma^1)^2 + \epsilon_2(\sigma^2)^2$ ($\epsilon_i = \pm 1$) and connection one-form σ_{12} are [24]:

$$\mathrm{d}\sigma^1 = \epsilon_1 \sigma_{12} \wedge \sigma^2, \qquad \mathrm{d}\sigma^2 = \epsilon_2 \sigma^1 \wedge \sigma_{12}, \qquad \mathrm{d}\sigma_{12} = -K\epsilon_1\epsilon_2 \sigma^1 \wedge \sigma^2, \tag{8}$$

in which K is the Gaussian curvature of S. One can easily check that, because of Eqs. (4), the one-forms defined in (a) satisfy Eq. (8) with $\epsilon_1 = 1, \epsilon_2 = K = -1$. The fact that $j^\infty(s)$ is I-generic implies that $\{j^\infty(s)^*\omega^2, j^\infty(s)^*\omega^3\}$ is a well-defined local moving coframe on the domain S of $u(x,t)$.

In the same way, if $j^\infty(s)$ is II-generic, one sees that the one-forms defined in (b) satisfy (8) with the same choices of $\epsilon_1, \epsilon_2, K$, and that $\{j^\infty(s)^*\omega^1, -j^\infty(s)^*\omega^3\}$ is a well-defined local moving coframe on S.

Finally, Eqs. (4) are identical to Eqs. (8) if one takes $\epsilon_1 = \epsilon_2 = 1$ and $K = -1$. Thus, if $j^\infty(s)$ is III-generic, the domain of $u(x,t)$ possesses the structure of a Riemannian pseudo-spherical surface equipped with the moving coframe $\{j^\infty(s)^*\omega^1, j^\infty(s)^*\omega^2\}$ and corresponding connection form $j^\infty(s)^*\omega^3$. □

Remark 1. If $j^\infty(s)$ is III-generic, say, Definition 1 implies that the graph of $j^\infty(s)$ is a submanifold of S^∞ possessing the structure of a pseudo-spherical surface with metric and connection form given in the (x,t) coordinates by $(\bar\omega^1)^2 + (\bar\omega^2)^2$ and $\bar\omega^3$, respectively, in which $\bar\omega^i := (j^\infty(s))^*\omega^i$, $i = 1, 2, 3$. This point of view is useful when studying (generalized) symmetries of equations of pseudo-spherical type [21].

Proposition 1 motivates the following definition, which will be used in Section 3.

Definition 3. Let $\Xi = 0$ be an equation of pseudo-spherical type with associated one-forms ω^i, $i = 1, 2, 3$. Then, (a) $\Xi = 0$ describes Riemannian pseudo-spherical surfaces if $\omega^1 \wedge \omega^2 \not\equiv 0$; (b) $\Xi = 0$ describes Lorentzian pseudo-spherical surfaces of type I if $\omega^2 \wedge \omega^3 \not\equiv 0$; and (c) $\Xi = 0$ describes Lorentzian pseudo-spherical surfaces of type II if $\omega^1 \wedge \omega^3 \not\equiv 0$.

Now consider the invariance properties of the structure equations (4). The following proposition holds.

Proposition 2. *Let ω^i, $i = 1, 2, 3$, be one-forms on $J^\infty E$. The structure equations*

$$d_H\omega^1 = \omega^3 \wedge \omega^2, \qquad d_H\omega^2 = \omega^1 \wedge \omega^3, \qquad d_H\omega^3 = \omega^1 \wedge \omega^2, \tag{9}$$

are invariant under the transformations

$$\hat\omega^1 = \omega^1 \cos\rho + \omega^2 \sin\rho, \qquad \hat\omega^2 = -\omega^1 \sin\rho + \omega^2 \cos\rho, \\ \hat\omega^3 = \omega^3 + d_H\rho, \tag{10}$$

$$\hat\omega^1 = \omega^1 \cosh\rho - \omega^3 \sinh\rho, \qquad \hat\omega^2 = \omega^2 + d_H\rho, \\ \hat\omega^3 = -\omega^1 \sinh\rho + \omega^3 \cosh\rho, \tag{11}$$

$$\hat\omega^1 = \omega^1 + d_H\rho, \qquad \hat\omega^2 = \omega^2 \cosh\rho + \omega^3 \sinh\rho, \\ \hat\omega^3 = \omega^2 \sinh\rho + \omega^3 \cosh\rho, \tag{12}$$

in which ρ is any smooth function on $J^\infty E$.

Let $\Xi = 0$ be a PSS equation with associated one-forms ω^i, $i = 1, 2, 3$, and let $j^\infty(s)$ be a solution of $\Xi = 0$. It follows from Eq. (3) and Propositions 1 and 2 that if $j^\infty(s)$ is III-generic, the pull-back of (10) by $j^\infty(s)$ is simply the transformation induced on the one-forms $j^\infty(s)^*\omega^i$ by a rotation of the moving orthonormal frame dual to the coframe $\{j^\infty(s)^*\omega^1, j^\infty(s)^*\omega^2\}$. Analogously, if $j^\infty(s)$ is II-generic the pull-back of (11) by $j^\infty(s)$ corresponds to a Lorentz boost of the moving frame dual to the coframe $\{j^\infty(s)^*\omega^1, -j^\infty(s)^*\omega^3\}$, and if $j^\infty(s)$ is I-generic the pull-back of (12) by $j^\infty(s)$ corresponds to a Lorentz boost of the frame dual to $\{j^\infty(s)^*\omega^2, j^\infty(s)^*\omega^3\}$.

Remark 2. A very interesting analysis of the invariance properties of (pseudo-)Riemannian surfaces of constant Gaussian curvature, and their relation with classical Bäcklund transformations, has been made by Crampin et al. in [6].

This section ends with an analysis of the close connection between equations of pseudo-spherical type and linear problems. It allows one to interpret (10)–(12) in terms of gauge transformations and, as already anticipated in Section 1, it helps to explain the appearance of pseudo-Riemannian manifolds in the theory.

Proposition 3. *Let $\Xi = 0$ be an equation of pseudo-spherical type with associated one-forms ω^i, $i = 1, 2, 3$. The equation $\Xi = 0$ is the integrability condition of the $sl(2, \mathbf{R})$-valued linear problem $\mathrm{d}v = \Omega v$, in which Ω is the one-form*

$$\Omega = \frac{1}{2}\begin{pmatrix} \omega^2 & \omega^1 - \omega^3 \\ \omega^1 + \omega^3 & -\omega^2 \end{pmatrix}, \tag{13}$$

i.e., $\mathrm{d}\Omega = \Omega \wedge \Omega$ whenever $u(x, t)$ is a local solution of $\Xi = 0$.

Thus, in the terminology of Crampin et al. [7], the one-form $\Omega(u(x, t))$ determines a “soliton connection” (see also [5,9]).

Example 2. Let ρ be a function on $J^\infty E$, $u(x, t)$ be a solution of $\Xi = 0$, and set $\hat{\rho} = \rho(u(x, t))$. The gauge transformation $\Omega(u(x, t)) \mapsto A\Omega(u(x, t))A^{-1} + \mathrm{d}AA^{-1}$, in which $\Omega(u(x, t))$ is defined by (13) and

$$A = \begin{pmatrix} \cosh\left(\frac{1}{2}\hat{\rho}\right) & \sinh\left(\frac{1}{2}\hat{\rho}\right) \\ \sinh\left(\frac{1}{2}\hat{\rho}\right) & \cosh\left(\frac{1}{2}\hat{\rho}\right) \end{pmatrix},$$

is precisely the pull-back by $u(x, t)$ of transformation (12).

The choice (13) is motivated by the relation between the one-forms ω^i, $i = 1, 2, 3$, associated to a PSS equation $\Xi = 0$, and the Maurer–Cartan structure equations of $SL(2, \mathbf{R})$.

Let X_A, $A = 1, 2, 3$, be a basis for the 2×2 matrix representation of $sl(2, \mathbf{R})$,

$$X_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad X_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad X_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{14}$$

Since $[X_1, X_2] = X_3$, $[X_2, X_3] = -X_1$, $[X_3, X_1] = X_2$, the non-zero structure constants C^I_{AB} are $C^3_{12} = 1$, $C^1_{23} = -1$, $C^2_{31} = 1$. Let μ^A, $A = 1, 2, 3$, be the basis of the right invariant

Maurer–Cartan forms of $SL(2,\mathbf{R})$ dual to $\{X_1, X_2, X_3\}$. The Maurer–Cartan equations $d\mu^I = \sum_{A<B} C^I_{AB}\mu^A \wedge \mu^B$ read

$$d\mu^1 = -\mu^2 \wedge \mu^3, \qquad d\mu^2 = -\mu^1 \wedge \mu^3, \qquad d\mu^3 = \mu^1 \wedge \mu^2, \tag{15}$$

and it follows that the $sl(2,\mathbf{R})$-valued one-form $\hat{\Omega} = \mu^1 X_1 + \mu^2 X_2 + \mu^3 X_3$, satisfies the zero curvature equation $d\hat{\Omega} = \hat{\Omega} \wedge \hat{\Omega}$. One now proves the following lemma.

Lemma 1. *Let $\Xi = 0$ be a PSS equation with associated one-forms ω^i, and let $u(x,t)$ be a solution of $\Xi = 0$ with domain M. For each $p \in M$ and $g \in SL(2,\mathbf{R})$, there exists a smooth map $F : U \subseteq M \to SL(2,\mathbf{R})$, in which U is open and $p \in U$, such that $F(p) = g$ and*

$$\omega^1(u(x,t)) = F^*(\mu^2), \qquad \omega^2(u(x,t)) = F^*(\mu^3), \qquad \omega^3(u(x,t)) = -F^*(\mu^1). \tag{16}$$

Thus, locally, the one-form Ω defined in (13) satisfies $\Omega(u(x,t)) = F^*\hat{\Omega}$.

Proof. Let π_1, π_2 be the projections from $M \times SL(2,\mathbf{R})$ onto M and $SL(2,\mathbf{R})$, respectively. Set $\bar{\omega}^i = \pi_1^*\omega^i(u(x,t))$, $\bar{\mu}^i = \pi_2^*\mu^i$, $i = 1, 2, 3$, and consider the ideal $\mathcal{I}$ of differential forms on $M \times SL(2,\mathbf{R})$ generated by the one-forms $\bar{\mu}^1 + \bar{\omega}^3$, $\bar{\mu}^2 - \bar{\omega}^1$, and $\bar{\mu}^3 - \bar{\omega}^2$. The structure equations (4) and (15) yield

$$\begin{aligned}
&d(\bar{\mu}^3 - \bar{\omega}^2) = \bar{\mu}^1 \wedge (\bar{\mu}^2 - \bar{\omega}^1) - \bar{\omega}^1 \wedge (\bar{\omega}^3 + \bar{\mu}^1),\\
&d(\bar{\mu}^2 - \bar{\omega}^1) = -\bar{\mu}^1 \wedge (\bar{\mu}^3 - \bar{\omega}^2) + \bar{\omega}^2 \wedge (\bar{\omega}^3 + \bar{\mu}^1),\\
&d(\bar{\mu}^1 + \bar{\omega}^3) = -\bar{\mu}^2 \wedge (\bar{\mu}^3 - \bar{\omega}^2) - \bar{\omega}^2 \wedge (\bar{\omega}^1 - \bar{\mu}^2),
\end{aligned}$$

and therefore $\mathcal{I}$ is closed under exterior differentiation. Since the Maurer–Cartan forms μ^i are linearly independent, the Frobenius theorem implies that there exists a unique maximal integral manifold of $\mathcal{I}$ through (p, g), and one can check (see [25, Theorem 2.34]) that this manifold is (locally) the graph of a function F satisfying $F(p) = g$ and (16). □

Conversely, assume now that there exists a linear problem

$$\frac{\partial v}{\partial x} = Xv, \qquad \frac{\partial v}{\partial t} = Tv, \tag{17}$$

in which X, T are $sl(2,\mathbf{R})$-valued functions on $J^\infty E$, such that the zero curvature condition

$$\frac{\partial X}{\partial t} - \frac{\partial T}{\partial x} + [X, T] = 0 \tag{18}$$

holds whenever $u(x,t)$ is a solution of the equation $\Xi = 0$. Set $\Omega = X\,dx + T\,dt$, and write Ω in the basis X_A given by (14) as $\Omega = e_1^A X_A\,dx + e_2^A X_A\,dt$. Define $\sigma^A = e_1^A\,dx + e_2^A\,dt$, $A = 1, 2, 3$, i.e.,

$$\begin{aligned}
&\sigma^1 = (-U_{21} + U_{12})\,dx + (-V_{21} + V_{12})\,dt, \quad \sigma^2 = (U_{21} + U_{12})\,dx + (V_{21} + V_{12})\,dt,\\
&\sigma^3 = 2U_{11}\,dx + 2V_{11}\,dt,
\end{aligned} \tag{19}$$

in which $X = (U_{\alpha\beta})$, $T = (V_{\alpha\beta})$. Eq. (18) implies that

$$\mathrm{d}\sigma^1 = -\sigma^2 \wedge \sigma^3, \qquad \mathrm{d}\sigma^2 = -\sigma^1 \wedge \sigma^3, \qquad \mathrm{d}\sigma^3 = \sigma^1 \wedge \sigma^2 \tag{20}$$

whenever $u(x,t)$ is a solution of $\Xi = 0$. One obtains different metric structures on the domain of a local solution $u(x,t)$ of $\Xi=0$ by choosing as connection form either $-\sigma^1(u(x,t))$, $\sigma^2(u(x,t))$, or $\sigma^3(u(x,t))$. This is the "splitting" result anticipated in Section 1.

Proposition 4. *Assume that the equation $\Xi = 0$ is the integrability condition of an $sl(2,\mathbf{R})$ valued linear problem $\mathrm{d}v = \Omega v$, in which $\Omega = X\,\mathrm{d}x + T\,\mathrm{d}t$, and consider the one-forms σ^i, $i = 1, 2, 3$, given by (19). Then,*

(a) *The one-forms $\omega^1 = \sigma^2$, $\omega^2 = \sigma^3$, and $\omega^3 = -\sigma^1$ satisfy the structure equations of a Riemannian surface of Gaussian curvature -1 whenever $u(x,t)$ is a solution of $\Xi = 0$. In particular, the equation $\Xi = 0$ is of pseudo-spherical type.*
(b) *The one-forms $\omega^1 = \sigma^3$, $\omega^2 = -\sigma^1$, and $\omega^3 = \sigma^2$ satisfy the structure equations of a Lorentzian surface of Gaussian curvature -1 whenever $u(x,t)$ is a solution of $\Xi = 0$.*
(c) *The one-forms $\omega^1 = \sigma^2$, $\omega^2 = \sigma^1$, and $\omega^3 = \sigma^3$ satisfy the structure equations of a Lorentzian surface of Gaussian curvature -1 whenever $u(x,t)$ is a solution of $\Xi = 0$.*

Remark 3. Proposition 4 follows from the structure equations (8) and (20). Note that it does not exhaust the metric structures one can impose on M starting from (17) and (18), essentially because $SL(2,\mathbf{R})$ appears also as isometry group of Lorentzian surfaces of positive constant Gaussian curvature. For instance, one can check that $\omega^1 = -\sigma^1$, $\omega^2 = \sigma^2$, $\omega^3 = -\sigma^3$ satisfy the structure equations of a Lorentzian surface of constant Gaussian curvature $+1$ whenever $u(x,t)$ is a solution of $\Xi = 0$. This observation will not be considered further here, since it does not yield new correspondence theorems. However, it may be of some physical interest (see [3,6,11]).

3. Correspondence theorems for PSS equations

Three correspondence theorems between appropriately generic solutions of scalar PSS equations $\Xi = 0$ and $\hat{\Xi} = 0$, are proved in this section. Pull-backs of one-forms ω by solutions of a PSS equation are denoted again by ω, no confusion should arise. Also, the space of independent variables of the equations $\Xi(x,t,u,\ldots) = 0$ and $\hat{\Xi}(\hat{x},\hat{t},\hat{u},\ldots) = 0$ are denoted by M and $\hat{M}$, respectively.

Remark 4. Suppose that one wishes to find a solution $\hat{u}(\hat{x},\hat{t})$ of the PSS equation $\hat{\Xi} = 0$ starting from a solution $u(x,t)$ of the PSS equation $\Xi = 0$. It will be always assumed that there exist one-forms $\hat{\omega}_0^i$, $i = 1, 2, 3$, associated to $\hat{\Xi} = 0$ such that this equation is not only sufficient but also necessary for the structure equation (4) to be satisfied. Such one-forms will be called target one-forms associated to $\hat{\Xi} = 0$ (see Section 4 for an example).

Lemma 2. *Let $\Xi(x,t,u,\ldots) = 0$ be a differential equation describing pseudo-spherical surfaces. There exists a gauge in which the one-forms $\omega^i = f_{i1}\,\mathrm{d}x + f_{i2}\,\mathrm{d}t$ associated to*

$\Xi = 0$ satisfy $\omega^i = u\,\mathrm{d}x + \beta\,\mathrm{d}t$ for at least one index i, $1 \le i \le 3$. Moreover, one can always choose associated one-forms ω^i such that $\omega^1 = u\,\mathrm{d}x + \beta\,\mathrm{d}t$.

Proof. Let $\omega^i = f_{i1}\,\mathrm{d}x + f_{i2}\,\mathrm{d}t$, $i = 1, 2, 3$, be one-forms associated with the PSS equation $\Xi = 0$. One can always assume that $f_{11} = 0$. In fact, if $f_{21} = 0$, one simply rotates the coframe using transformation (10) with $\rho = \pi/2$ to obtain associated one-forms $\hat{\omega}^i = \hat{f}_{i1}\,\mathrm{d}x + \hat{f}_{i2}\,\mathrm{d}t$ satisfying $\hat{f}_{11} = 0$. If $f_{21} \neq 0$, one applies transformation (10) with $\rho = \arctan(-f_{11}/f_{21})$ to find again $\hat{f}_{11} = 0$.

Assume then that $\omega^i = f_{i1}\,\mathrm{d}x + f_{i2}\,\mathrm{d}t$, $i = 1, 2, 3$, satisfy $f_{11} = 0$. There are three cases. First, if $f_{31} \neq 0$, one applies transformation (11) with $\rho = \operatorname{arcsinh}(-u/f_{31})$ to find new associated one-forms with $\hat{f}_{11} = u$. Second, if $f_{31} = 0$ but $f_{21} \neq 0$, one uses transformation (12) with $\rho = \operatorname{arcsinh}(-u/f_{21})$ to obtain $\hat{f}_{31} = u$. Third, if $f_{31} = f_{21} = 0$, one uses (11) to find new associated one-forms with $\hat{f}_{11} = \hat{f}_{31} = 0$ and $\hat{f}_{21} \neq 0$, and proceed as in the second case.

The last assertion of the lemma is trivial if the construction above yields either $\hat{f}_{11} = u$ or $\hat{f}_{21} = u$. Suppose then that the equation $\Xi = 0$ has associated one-forms ω^i satisfying $\omega^3 = u\,\mathrm{d}x + \beta\,\mathrm{d}t$. Reasoning as in the first paragraph of this proof, one gets new associated one-forms $\hat{\omega}^i$ satisfying either

$$\hat{f}_{11} = 0 \quad \text{and} \quad \hat{f}_{31} = u, \quad \text{or} \quad \hat{f}_{11} = 0 \quad \text{and} \quad \hat{f}_{31} = u + \mathrm{D}_x\left(\arctan\left(-\frac{f_{11}}{f_{21}}\right)\right).$$

In any of these two cases $\hat{f}_{31} \neq 0$, and therefore, reasoning as in the second paragraph, one can find new one-forms, $\hat{\omega}^i_{\mathrm{new}}$ say, satisfying $\hat{f}_{11\,\mathrm{new}} = u$, as claimed. □

The first correspondence result of this section is a slight generalization of Kamran and Tenenblat's theorem [13].

Theorem 1. *Let $\Xi(x, t, u, \ldots) = 0$ and $\hat{\Xi}(\hat{x}, \hat{t}, \hat{u}, \ldots) = 0$ be two scalar equations describing Riemannian pseudo-spherical surfaces. Any III-generic solution $u(x, t)$ of $\Xi(x, t, u, \ldots) = 0$ determines a III-generic solution $\hat{u}(\hat{x}, \hat{t})$ of $\hat{\Xi}(\hat{x}, \hat{t}, \hat{u}, \ldots) = 0$.*

Proof. Choose associated one-forms ω^i, $i = 1, 2, 3$, of $\Xi = 0$ and let $\hat{\omega}^i_0$, $i = 1, 2, 3$, be target one-forms associated to the PSS equation $\hat{\Xi} = 0$. Apply Lemma 2 to find one-forms $\hat{\omega}^i = \hat{f}_{i1}\,\mathrm{d}\hat{x} + \hat{f}_{i2}\,\mathrm{d}\hat{t}$ associated to $\hat{\Xi} = 0$ such that $\hat{\omega}^1 = \hat{u}\,\mathrm{d}\hat{x} + \beta\,\mathrm{d}\hat{t}$ for some function β depending on $\hat{u}$ and a finite number of its derivatives. Following Kamran and Tenenblat [13], one obtains the formula

$$\hat{u} \circ \Phi = \frac{1}{J}\left[(\cos\theta f_{11} + \sin\theta f_{21})\frac{\partial\psi}{\partial t} - (\cos\theta f_{12} + \sin\theta f_{22})\frac{\partial\psi}{\partial x}\right], \tag{21}$$

in which $\Phi(x, t) = (\varphi(x, t), \psi(x, t))$ is a local diffeomorphism with Jacobian J, and the functions $\theta(x, t)$, $\varphi(x, t)$, and $\psi(x, t)$ are smooth solutions of the system of equations

$$(\Phi^* \hat{f}_{21})\frac{\partial\varphi}{\partial x} + (\Phi^* \hat{f}_{22})\frac{\partial\psi}{\partial x} = -\sin\theta f_{11} + \cos\theta f_{21}, \tag{22}$$

$$(\Phi^*\hat{f}_{21})\frac{\partial\varphi}{\partial t} + (\Phi^*\hat{f}_{22})\frac{\partial\psi}{\partial t} = -\sin\theta f_{12} + \cos\theta f_{22}, \tag{23}$$

$$(\Phi^*\hat{f}_{31})\frac{\partial\varphi}{\partial x} + (\Phi^*\hat{f}_{32})\frac{\partial\psi}{\partial x} = f_{31} + \frac{\partial\theta}{\partial x}, \tag{24}$$

$$(\Phi^*\hat{f}_{31})\frac{\partial\varphi}{\partial t} + (\Phi^*\hat{f}_{32})\frac{\partial\psi}{\partial t} = f_{32} + \frac{\partial\theta}{\partial t}, \tag{25}$$

$$-(\Phi^*\hat{f}_{12})J = (\cos\theta f_{11} + \sin\theta f_{21})\frac{\partial\varphi}{\partial t} - (\cos\theta f_{12} + \sin\theta f_{22})\frac{\partial\varphi}{\partial x}, \tag{26}$$

where the pull-backs of $\hat{u}$ and its derivatives with respect to $\hat{x}, \hat{t}$ appearing in the functions $(\Phi^*\hat{f}_{ij})(x,t), i = 1,2,3, j = 1,2$, have been evaluated by means of (21) and its derivatives. Eq. (21) determines a solution $\hat{u}(\hat{x},\hat{t})$ of $\hat{\Xi}(\hat{x},\hat{t},\hat{u},\dots) = 0$. A particularly clear way to see this is the following: consider the connections $\hat{\Omega}$ and $\hat{\Omega}_0$ determined by the one-forms $\hat{\omega}^i$ and $\hat{\omega}^i_0$, respectively. By construction, there exists an $SL(2,\mathbf{R})$-valued matrix A whose entries are smooth functions on $J^\infty E$ such that

$$\hat{\Omega} = A\hat{\Omega}_0 A^{-1} + \mathrm{d}AA^{-1}, \qquad \hat{\Theta} = A\hat{\Theta}_0 A^{-1},$$

in which $\hat{\Theta}$ and $\hat{\Theta}_0$ are the curvatures of $\hat{\Omega}$ and $\hat{\Omega}_0$, respectively. The fact that $\Phi(x,t)$ is a local diffeomorphism, together with Eqs. (21)–(26), implies the structure equations

$$\mathrm{d}\hat{\omega}^1 = \hat{\omega}^3 \wedge \hat{\omega}^2, \qquad \mathrm{d}\hat{\omega}^2 = \hat{\omega}^1 \wedge \hat{\omega}^3, \qquad \mathrm{d}\hat{\omega}^3 = \hat{\omega}^1 \wedge \hat{\omega}^2, \tag{27}$$

so that $\hat{\Theta} = 0$. It follows that $\hat{\Theta}_0 = 0$, and, therefore, the target one-forms $\hat{\omega}^i_0$ also satisfy (27). This means that (21) determines a solution of $\hat{\Xi} = 0$, as claimed. That $\hat{u}$ is III-generic (with respect to the associated one-forms $\hat{\omega}^i$) follows from the equation $\Phi^*(\hat{\omega}^1 \wedge \hat{\omega}^2) = \omega^1 \wedge \omega^2$, an easy consequence of (22)–(26). □

Remark 5. Kamran and Tenenblat's theorem was proved in [13] under the hypothesis (notation as in the proof of Theorem 1) $\hat{f}_{ij} = \hat{G}(\hat{u})$ and $\hat{G}'(\hat{u}) \neq 0$ for at least one $\hat{f}_{ij}$, $1 \leq i \leq 3$, $1 \leq j \leq 2$. Lemma 2 and the above argument show that one does not need to make this a priori assumption.

Example 3 (Non-linear superposition of sine-Gordon solutions). The sine-Gordon equation

$$\frac{\partial^2\theta}{\partial u^2} - \frac{\partial^2\theta}{\partial v^2} = \sin\theta\,\cos\theta \tag{28}$$

describes pseudo-spherical surfaces with associated functions

$$f_{11} = 0, \qquad f_{12} = \sin\theta, \qquad f_{21} = -\cos\theta, \qquad f_{22} = 0,$$
$$f_{31} = \theta_v, \qquad f_{32} = \theta_u. \tag{29}$$

Let $\theta(u,v)$ be a III-generic solution of Eq. (28). Theorem 1 allows one to obtain the traveling wave solution of Burgers equation from (28) and (29) as follows.

Consider a function $\omega(u, v)$ determined by the completely integrable Pfaffian system

$$\theta_v + \omega_u = -\sin\omega\,\cos\theta, \qquad \theta_u + \omega_v = \cos\omega\,\sin\theta. \tag{30}$$

The function $\omega(u, v)$ is also a solution of Eq. (28); in fact, Eqs. (30) are simply the equations determining the Bianchi transform of the pseudo-spherical surface described by the one-forms (29) [8]. Now introduce potentials $\xi(u, v)$ and $\zeta(u, v)$ thus

$$d\xi = -(\cos\omega\,\cos\theta\,du + \sin\omega\,\sin\theta\,dv), \tag{31}$$

$$d\zeta = \exp(\xi)(\cos\theta\,\sin\omega\,du - \sin\theta\,\cos\omega\,dv). \tag{32}$$

The functions ξ and ζ are well-defined because of (30). Note also that the map $\chi : (u, v) \mapsto (\zeta(u, v), \xi(u, v))$ is a local diffeomorphism because $\theta(u, v)$ is III-generic, and that

$$\chi^*(d\xi^2 + \exp(-2\xi)\,d\zeta^2) = \cos^2\theta\,du^2 + \sin^2\theta\,dv^2.$$

Thus, χ determines a local isometry between the pseudo-spherical surface described by the one-forms (29) and the standard pseudo-sphere.

Now, Burgers' equation

$$\hat{u}_{\hat{t}} = \hat{u}_{\hat{x}\hat{x}} - 2\hat{u}\hat{u}_{\hat{x}} \tag{33}$$

describes pseudo-spherical surfaces with associated functions

$$\hat{f}_{11} = \hat{u}, \qquad \hat{f}_{12} = -\hat{u}^2 + \hat{u}_{\hat{x}}, \qquad \hat{f}_{21} = 1, \qquad \hat{f}_{22} = -\hat{u},$$
$$\hat{f}_{31} = 1, \qquad \hat{f}_{32} = -\hat{u}. \tag{34}$$

Define the transformation

$$\hat{x} = \varphi(u, v) = 1 - \zeta(u, v) - \exp(\xi(u, v)), \tag{35}$$

$$\hat{t} = \psi(u, v) = \ln|1 - \exp(\xi(u, v))| - 1 + \zeta(u, v) + \exp(\xi(u, v)). \tag{36}$$

One can check that, if $\tilde{\theta} = \omega(u, v) - \pi/2$, the map $\Phi : (u, v) \mapsto (\hat{x}, \hat{t})$ given by (35) and (36) is a local diffeomorphism which satisfies the system of Eqs. (22)–(26) (with "θ" replaced by "$\tilde{\theta}$"). Substituting into Eq. (21) (again, with "θ", replaced by "$\tilde{\theta}$") one finds that the pull-back of $\hat{u}(\hat{x}, \hat{t})$ by Φ is

$$\hat{u} \circ \Phi = \frac{1 - \exp(\xi(u, v))}{\exp(\xi(u, v))},$$

and, therefore, one finally obtains

$$\hat{u}(\hat{x}, \hat{t}) = \frac{\exp(\hat{x} + \hat{t})}{1 - \exp(\hat{x} + \hat{t})}, \tag{37}$$

the traveling wave solution of Eq. (33).

The crucial Eqs. (22)–(26) appearing in the proof of the last theorem were found in [13] with the aid of transformation (10) of Proposition 2. Transformations (11) and (12) are now used to find correspondence results analogous to Theorem 1, this time by considering equations describing Lorentzian pseudo-spherical surfaces of types I and II. The first such (Theorem 3) follows from Theorem 2 and Lemma 3.

Theorem 2. *Let $\Xi(x, t, u, \dots) = 0$ and $\hat{\Xi}(\hat{x}, \hat{t}, \hat{u}, \dots) = 0$ be two equations describing Lorentzian pseudo-spherical surfaces of type II with associated one-forms $\omega^i = f_{i1}\,dx + f_{i2}\,dt$ and $\hat{\omega}^i = \hat{f}_{i1}\,d\hat{x} + \hat{f}_{i2}\,d\hat{t}$, $i = 1, 2, 3$, respectively. For any II-generic solutions $u(x, t)$ of $\Xi = 0$ and $\hat{u}(\hat{x}, \hat{t})$ of $\hat{\Xi} = 0$, there exist a local diffeomorphism $\Upsilon : V \to \hat{V}$, in which V and $\hat{V}$ are open subsets of the domains of $u(x, t)$ and $\hat{u}(\hat{x}, \hat{t})$, respectively, and a smooth function $\mu : V \to \mathbf{R}$, such that the pull-backs of ω^i by $u(x, t)$, and of $\hat{\omega}^i$ by $\hat{u}(\hat{x}, \hat{t})$, satisfy*

$$\Upsilon^*\hat{\omega}^1 = \omega^1 \cosh\mu - \omega^3 \sinh\mu, \tag{38}$$

$$\Upsilon^*\hat{\omega}^2 = \omega^2 + d\mu, \tag{39}$$

$$\Upsilon^*\hat{\omega}^3 = -\omega^1 \sinh\mu + \omega^3 \cosh\mu. \tag{40}$$

Proof. Define one-forms $\hat{\sigma}^i$, $i = 1, 2, 3$, on an open subset $\hat{W}$ of $\hat{V}$ by

$$\hat{\sigma}^1 = \frac{1}{\hat{x}}\,d\hat{x}, \qquad \hat{\sigma}^2 = \frac{1}{\hat{x}}\,d\hat{t}, \qquad \hat{\sigma}^3 = \frac{1}{\hat{x}}\,d\hat{t}. \tag{41}$$

The structure equations (8) with $\epsilon_1 = 1$ and $\epsilon_2 = -1$ imply that $(\hat{\sigma}^1)^2 - (\hat{\sigma}^2)^2$ is a Lorentz metric of constant Gaussian curvature $K = -1$ on $\hat{W}$, and that $\hat{\sigma}^3$ is the connection one-form corresponding to the moving coframe $\{\hat{\sigma}^1, \hat{\sigma}^2\}$. Let $u(x, t)$ be a II-generic solution of the equation $\Xi = 0$, and consider the pull-backs of the one-forms ω^i, $i = 1, 2, 3$, by $u(x, t)$. Motivated by Propositions 1 and 2, one claims that there exists a function $\Gamma : V \to \hat{W}$, $\Gamma(x, t) = (\alpha(x, t), \beta(x, t))$, and a real-valued function $\theta(x, t)$ on V such that

$$\Gamma^*\hat{\sigma}^1 = \omega^1 \cosh\theta - \omega^3 \sinh\theta, \tag{42}$$

$$\Gamma^*\hat{\sigma}^2 = \omega^1 \sinh\theta - \omega^3 \cosh\theta, \tag{43}$$

$$\Gamma^*\hat{\sigma}^3 = \omega^2 + d\theta. \tag{44}$$

Indeed, note that

$$\Gamma^*\hat{\sigma}^1 = \frac{1}{\alpha}\,d\alpha, \qquad \Gamma^*\hat{\sigma}^2 = \frac{1}{\alpha}\,d\beta, \qquad \Gamma^*\hat{\sigma}^3 = \frac{1}{\alpha}\,d\beta,$$

so that Eqs. (42)–(44) are equivalent to the Pfaffian system

$$\alpha_x = \alpha(f_{11} \cosh\theta - f_{31} \sinh\theta), \qquad \alpha_t = \alpha(f_{12} \cosh\theta - f_{32} \sinh\theta), \tag{45}$$

$$\beta_x = \alpha(f_{11} \sinh\theta - f_{31} \cosh\theta), \qquad \beta_t = \alpha(f_{12} \sinh\theta - f_{32} \cosh\theta), \tag{46}$$

$$\theta_x = f_{11} \sinh\theta - f_{31} \cosh\theta - f_{21}, \qquad \theta_t = f_{12} \sinh\theta - f_{32} \cosh\theta - f_{22}. \tag{47}$$

It is not difficult to see that this system is completely integrable for $\alpha(x, t)$, $\beta(x, t)$ and $\theta(x, t)$, since the pulled-back one-forms ω^i, $i = 1, 2, 3$, satisfy the structure equations of a pseudo-spherical surface. Moreover, $\Gamma = (\alpha, \beta)$ is a local diffeomorphism, since

$$\alpha_x \beta_t - \alpha_t \beta_x = -\alpha^2 (f_{11} f_{32} - f_{31} f_{12}),$$

and $f_{11} f_{32} - f_{31} f_{12} \neq 0$ because $u(x, t)$ is a II-generic solution of $\Xi = 0$.

Now, one can use the same argument as above to find a local diffeomorphism $\tilde{\Gamma} : \hat{V} \to \hat{W}$ and a function $\tilde{\theta} : \hat{V} \to \mathbf{R}$ such that

$$\tilde{\Gamma}^* \hat{\sigma}^1 = \hat{\omega}^1 \cosh \tilde{\theta} - \hat{\omega}^3 \sinh \tilde{\theta}, \qquad \tilde{\Gamma}^* \hat{\sigma}^2 = \hat{\omega}^1 \sinh \tilde{\theta} - \hat{\omega}^3 \cosh \tilde{\theta},$$
$$\tilde{\Gamma}^* \hat{\sigma}^3 = \hat{\omega}^2 + \mathrm{d}\tilde{\theta}.$$

It is then straightforward to check that Eqs. (38)–(40) are satisfied if one defines

$$\Upsilon = \tilde{\Gamma}^{-1} \circ \Gamma, \qquad \mu = \theta - \tilde{\theta} \circ \Upsilon. \qquad \square$$

In geometrical terms, Theorem 2 says that there exists a local isometry Υ between the Lorentzian surfaces $(V, (\omega^1)^2 - (-\omega^3)^2)$ and $(\hat{V}, (\hat{\omega}^1)^2 - (-\hat{\omega}^3)^2)$ determined by the II-generic solutions $u(x,t)$ of $\Xi = 0$ and $\hat{u}(\hat{x},\hat{t})$ of $\hat{\Xi} = 0$, respectively, and that Υ preserves the moving frames $\{\omega^1, -\omega^3\}$ and $\{\hat{\omega}^1, -\hat{\omega}^3\}$ up to a Lorentz boost.

Next, one uses Theorem 2 to construct a Bäcklund-like transformation from $u(x,t)$ to $\hat{u}(\hat{x},\hat{t})$. For this, one needs the following lemma.

Lemma 3. *Let $\Xi(x,t,u,\ldots) = 0$ and $\hat{\Xi}(\hat{x},\hat{t},\hat{u},\ldots) = 0$ be two scalar equations describing Lorentzian pseudo-spherical surfaces of type II with associated one-forms $\omega^i = f_{i1}\,\mathrm{d}x + f_{i2}\,\mathrm{d}t$ and $\hat{\omega}^i = \hat{f}_{i1}\,\mathrm{d}\hat{x} + \hat{f}_{i2}\,\mathrm{d}\hat{t}$, $i = 1,2,3$, respectively, in which $\hat{f}_{21} = \hat{u}$. Let $\Upsilon(x,t) = (\phi(x,t), \psi(x,t))$ be a smooth map with Jacobian $J = \phi_x\psi_t - \phi_t\psi_x$ from (an open subset of) M to (an open subset of) $\hat{M}$, and let μ be a smooth map from (an open subset of) M to R. The system of equations*

$$(\Upsilon^*\hat{f}_{11})\phi_x + (\Upsilon^*\hat{f}_{12})\,\psi_x = f_{11}\cosh\mu - f_{31}\sinh\mu, \tag{48}$$

$$(\Upsilon^*\hat{f}_{11})\phi_t + (\Upsilon^*\hat{f}_{12})\psi_t = f_{12}\cosh\mu - f_{32}\sinh\mu, \tag{49}$$

$$J(\Upsilon^*\hat{f}_{22}) = -[\phi_t(f_{21} + \mu_x) - \phi_x(f_{22} + \mu_t)], \tag{50}$$

$$(\Upsilon^*\hat{f}_{31})\phi_x + (\Upsilon^*\hat{f}_{32})\psi_x = -f_{11}\sinh\mu + f_{31}\cosh\mu, \tag{51}$$

$$(\Upsilon^*\hat{f}_{31})\phi_t + (\Upsilon^*\hat{f}_{32})\psi_t = -f_{12}\sinh\mu + f_{32}\cosh\mu, \tag{52}$$

in which the pull-backs of $\hat{u}$ and its derivatives with respect to $\hat{x}, \hat{t}$ appearing in the functions $(\Upsilon^*\hat{f}_{ij})(x,t)$ have been evaluated by means of the equation

$$\hat{u} \circ \Upsilon = \frac{1}{J}(\psi_t(f_{21} + \mu_x) - \psi_x(f_{22} + \mu_t)), \tag{53}$$

admits—whenever $u(x,t)$ is a II-generic solution of $\Xi = 0$—a local solution $\phi(x,t)$, $\psi(x,t)$, $\mu(x,t)$ such that $\Upsilon(x,t) = (\phi(x,t), \psi(x,t))$ is a local diffeomorphism.

Proof. In order to see that Eqs. (48)–(52) have local solutions with the properties listed in the lemma, take *any* II-generic solution of $\hat{\Xi} = 0$, $\hat{u}(\hat{x},\hat{t})$ say, and identify it with an holonomic local section $j^\infty(\hat{s})$ of the equation manifold $\hat{S}^\infty$ of $\hat{\Xi} = 0$. By Theorem 2, there exist functions ϕ, ψ and μ satisfying the first-order system of equations

$$((j^\infty(\hat{s})^*\hat{f}_{11}) \circ \Upsilon)\phi_x + ((j^\infty(\hat{s})^*\hat{f}_{12}) \circ \Upsilon)\psi_x = f_{11}\cosh\mu - f_{31}\sinh\mu, \tag{54}$$

$$((j^\infty(\hat{s})^*\hat{f}_{11}) \circ \Upsilon)\phi_t + ((j^\infty(\hat{s})^*\hat{f}_{12}) \circ \Upsilon)\psi_t = f_{12}\cosh\mu - f_{32}\sinh\mu, \tag{55}$$

$$((j^\infty(\hat{s})^* \hat{f}_{31}) \circ \Upsilon)\phi_x + ((j^\infty(\hat{s})^* \hat{f}_{32}) \circ \Upsilon)\psi_x = -f_{11} \sinh\mu + f_{31} \cosh\mu, \tag{56}$$

$$((j^\infty(\hat{s})^* \hat{f}_{31}) \circ \Upsilon)\phi_t + ((j^\infty(\hat{s})^* \hat{f}_{32}) \circ \Upsilon)\psi_t = -f_{12} \sinh\mu + f_{32} \cosh\mu, \tag{57}$$

$$(\hat{u} \circ \Upsilon)\phi_x + ((j^\infty(\hat{s})^* \hat{f}_{22}) \circ \Upsilon)\psi_x = f_{21} + \mu_x, \tag{58}$$

$$(\hat{u} \circ \Upsilon)\phi_t + ((j^\infty(\hat{s})^* \hat{f}_{22}) \circ \Upsilon)\psi_t = f_{22} + \mu_t, \tag{59}$$

in which $\Upsilon(x,t) = (\phi(x,t), \psi(x,t))$, and such that Υ is a local diffeomorphism. These functions ϕ, ψ and μ also satisfy the system of Eqs. (48)–(52). Indeed, Eqs. (58) and (59) can be considered as a linear system for $\hat{u} \circ \Upsilon$ and $(j^\infty(\hat{s})^* \hat{f}_{22}) \circ \Upsilon$, and solution of this system yields Eq. (53) and

$$J[(j^\infty(\hat{s})^* \hat{f}_{22}) \circ \Upsilon] = -[\phi_t(f_{21} + \mu_x) - \phi_x(f_{22} + \mu_t)]. \tag{60}$$

Now, let $\hat{g}_{ij}$, $i = 1, 2, 3$, $j = 1, 2$, be the functions depending on x, t, ϕ, ψ, and their derivatives, which are obtained from $\hat{f}_{ij}$ by computing the pull-backs $\Upsilon^* \hat{f}_{ij}$ as in the enunciate of the lemma. Then, on the solutions $\phi(x,t)$, $\psi(x,t)$, $\mu(x,t)$ of the system (54)–(59), one has, for any i and j,

$$(\hat{g}_{ij})(x,t) = (j^\infty(\hat{s})^* \hat{f}_{ij}) \circ \Upsilon(x,t),$$

since on these functions ϕ, ψ, μ, Eq. (53) is an identity.

Thus, the system (48)–(52) reduces to the first-order system (54)–(57) and (60), and the result follows. □

Theorem 2 and Lemma 3 allow one to prove the following correspondence result.

Theorem 3. *Let $\Xi(x,t,u,\ldots) = 0$ and $\hat{\Xi}(\hat{x},\hat{t},\hat{u},\ldots) = 0$ be two scalar equations describing Lorentzian pseudo-spherical surfaces of type II, and assume that $\Xi = 0$ has associated one-forms $\omega^i = f_{i1}\,\mathrm{d}x + f_{i2}\,\mathrm{d}t$, $i = 1, 2, 3$. Any II-generic solution $u(x,t)$ of $\Xi(x,t,u,\ldots) = 0$ determines a II-generic solution $\hat{u}(\hat{x},\hat{t})$ of $\hat{\Xi}(\hat{x},\hat{t},\hat{u},\ldots) = 0$ by*

$$\hat{u} \circ \Upsilon = \frac{1}{J}(\psi_t(f_{21} + \mu_x) - \psi_x(f_{22} + \mu_t)), \tag{61}$$

in which $\Upsilon(x,t) = (\phi(x,t), \psi(x,t))$, and ϕ, ψ, μ are given by (48)–(52).

Proof. Let $\hat{\omega}_0^i$, $i = 1, 2, 3$, be target one-forms associated to $\hat{\Xi} = 0$. Applying a gauge transformation as in Lemma 2, one obtains one-forms $\hat{\omega}^i = \hat{f}_{i1} + \hat{f}_{i2}$ associated with the equation $\hat{\Xi} = 0$ such that $\hat{f}_{21} = \hat{u}$. Fix a II-generic solution $u(x,t)$ of equation $\Xi(x,t,u,\ldots) = 0$, and consider the system of Eqs. (48)–(52). By Lemma 3, this system possesses local solutions ϕ, ψ and μ such that $\Upsilon = (\phi, \psi)$ is a local diffeomorphism. One then defines $\hat{u} \circ \Upsilon$ by Eq. (61), thereby obtaining a system of six equations equivalent to

$$\Upsilon^* \hat{\omega}^1 = \omega^1 \cosh\mu - \omega^3 \sinh\mu, \tag{62}$$

$$\Upsilon^* \hat{\omega}^2 = \omega^2 + \mathrm{d}\mu, \tag{63}$$

$$\Upsilon^* \hat{\omega}^3 = -\omega^1 \sinh\mu + \omega^3 \cosh\mu. \tag{64}$$

Since Υ is a local diffeomorphism, and the one-forms ω^i satisfy the structure equations (4) of a pseudo-spherical surface, so do the one-forms $\hat{\omega}^i$. It follows that the target one-forms $\hat{\omega}_0^i$ satisfy

$$\mathrm{d}\hat{\omega}_0^1 = \hat{\omega}_0^3 \wedge \hat{\omega}_0^2, \qquad \mathrm{d}\hat{\omega}_0^2 = \hat{\omega}_0^1 \wedge \hat{\omega}_0^3, \qquad \mathrm{d}\hat{\omega}_0^3 = \hat{\omega}_0^1 \wedge \hat{\omega}_0^2. \tag{65}$$

This means that (61) determines a solution of $\hat{\Xi} = 0$, as claimed. Finally, note that if one multiplies Eqs. (48) and (52), then multiplies Eqs. (49) and (51), and subtract the results, one obtains the identity

$$[(\hat{f}_{11} \circ \Upsilon)(\hat{f}_{32} \circ \Upsilon) - (\hat{f}_{12} \circ \Upsilon)(\hat{f}_{31} \circ \Upsilon)][\phi_x \psi_t - \phi_t \psi_x] = f_{11} f_{32} - f_{12} f_{31}, \tag{66}$$

so that $\hat{u}(\hat{x}, \hat{t})$ is a II-generic solution of the equation $\hat{\Xi} = 0$. □

Example 4 (Straightening-out elliptic sine- and sinh-Gordon solutions). The elliptic sine-Gordon equation

$$\frac{\partial^2 \theta}{\partial u^2} + \frac{\partial^2 \theta}{\partial v^2} = \sin\theta \, \cos\theta \tag{67}$$

is usually considered in pseudo-Riemannian geometry (see for instance [6,11,17,24]). It describes pseudo-spherical surfaces with associated functions f_{ij} given by

$$f_{11} = \cos\theta, \qquad f_{12} = 0, \qquad f_{21} = \theta_v, \qquad f_{22} = -\theta_u,$$
$$f_{31} = 0, \qquad f_{32} = -\sin\theta. \tag{68}$$

Let $\theta(u, v)$ be a II-generic solution of (67). The aim of this example is to show that one can use Theorem 3 to obtain a solution of the linear second-order equation

$$\hat{u}_{\hat{t}} = \hat{u}_{\hat{x}\hat{x}} + \hat{u}_{\hat{x}} \tag{69}$$

from (67) and (68).

First, consider a function $\omega(u, v)$ given by the completely integrable Pfaffian system

$$\theta_v + \omega_u = \cos\theta \, \sinh\omega, \qquad -\theta_u + \omega_v = \sin\theta \, \cosh\omega. \tag{70}$$

Note that the function $\omega(u, v)$ satisfies the equation

$$\frac{\partial^2 \omega}{\partial u^2} + \frac{\partial^2 \omega}{\partial v^2} = \sinh\omega \, \cosh\omega, \tag{71}$$

so that Eqs. (70) determine a Bäcklund-like transformation between the elliptic sine-Gordon and the elliptic sinh-Gordon equations. The system (70) has been obtained here with the aid of Eqs. (47) appearing in the proof of Theorem 2, other derivations are in [6,17] (see also [24] for a related result).

Using θ and ω, one introduces potentials $\xi(u, v)$ and $\beta(u, v)$ thus

$$\mathrm{d}\xi = \cos\theta \, \cosh\omega \, \mathrm{d}u + \sin\theta \, \sinh\omega \, \mathrm{d}v, \tag{72}$$

$$\mathrm{d}\beta = \exp(\xi)(\cos\theta \, \sinh\omega \, \mathrm{d}u + \sin\theta \, \cosh\omega \, \mathrm{d}v). \tag{73}$$

The functions ξ and β are well-defined because of (67) and (70).

Now, the linear Eq. (69) describes pseudo-spherical surfaces with associated functions $\hat{f}_{ij}$ given by

$$\hat{f}_{11} = 1, \qquad \hat{f}_{12} = 0, \qquad \hat{f}_{21} = \hat{u}, \qquad \hat{f}_{22} = \hat{u}_{\hat{x}}, \\ \hat{f}_{31} = -\hat{u}, \qquad \hat{f}_{32} = -\hat{u}_{\hat{x}}. \tag{74}$$

Define the map $\Upsilon : (u, v) \to (\hat{x}, \hat{t})$ by means of

$$\hat{x} = \phi(u, v) = \xi(u, v), \tag{75}$$

$$\hat{t} = \psi(u, v) = 1 - \xi(u, v) + \beta(u, v) \exp(-\xi(u, v)). \tag{76}$$

One can check that Υ is a local diffeomorphism which satisfies the system of Eqs. (48)–(52) if one sets $\mu(u, v) = \omega(u, v)$. Formula (61) then yields

$$\hat{u} \circ \Upsilon = 1 + \beta(u, v) \exp(-\xi(u, v)),$$

and, therefore, one obtains the following solution of (69):

$$\hat{u}(\hat{x}, \hat{t}) = \hat{x} + \hat{t}.$$

The correspondence results for equations describing Lorentzian pseudo-spherical surfaces of type I are stated below. Their proofs are modeled after the ones already given, and are therefore omitted.

Instead of Theorem 2 one finds the following.

Theorem 4. *Let $\Xi(x, t, u, \ldots) = 0$ and $\hat{\Xi}(\hat{x}, \hat{t}, \hat{u}, \ldots) = 0$ be two equations describing Lorentzian pseudo-spherical surfaces of type I with associated one-forms $\omega^i = f_{i1}\, \mathrm{d}x + f_{i2}\, \mathrm{d}t$ and $\hat{\omega}^i = \hat{f}_{i1}\, \mathrm{d}\hat{x} + \hat{f}_{i2}\, \mathrm{d}\hat{t}$, $i = 1, 2, 3$, respectively. For any I-generic solutions $u(x, t)$ of $\Xi = 0$ and $\hat{u}(\hat{x}, \hat{t})$ of $\hat{\Xi} = 0$, there exist a local diffeomorphism $\Psi : V \to \hat{V}$, in which V and $\hat{V}$ are open subsets of the domains of $u(x, t)$ and $\hat{u}(\hat{x}, \hat{t})$*, respectively, and a smooth function $\nu : V \to \mathbf{R}$, *such that the pull-backs of ω^i* by $u(x, t)$ *and of $\hat{\omega}^i$ by $\hat{u}(\hat{x}, \hat{t})$ satisfy*

$$\Psi^* \hat{\omega}^1 = \omega^1 + \mathrm{d}\nu, \tag{77}$$

$$\Psi^* \hat{\omega}^2 = \omega^2 \cosh \nu + \omega^3 \sinh \nu, \tag{78}$$

$$\Psi^* \hat{\omega}^3 = \omega^2 \sinh \nu + \omega^3 \cosh \nu. \tag{79}$$

The technical Lemma 3 now becomes the following.

Lemma 4. *Let $\Xi(x, t, u, \ldots) = 0$ and $\hat{\Xi}(\hat{x}, \hat{t}, \hat{u}, \ldots) = 0$ be two scalar equations describing Lorentzian pseudo-spherical surfaces of type I with associated one-forms $\omega^i = f_{i1}\, \mathrm{d}x + f_{i2}\, \mathrm{d}t$ and $\hat{\omega}^i = \hat{f}_{i1}\, \mathrm{d}\hat{x} + \hat{f}_{i2}\, \mathrm{d}\hat{t}$, $i = 1, 2, 3$, respectively, in which $\hat{f}_{11} = \hat{u}$. Let $\Psi(x, t) = (\gamma(x, t), \delta(x, t))$ be a smooth map with Jacobian $J = \gamma_x \delta_t - \gamma_t \delta_x$ from (an open subset of) M to (an open subset of) $\hat{M}$, and let ν be a smooth map from (an open subset of) M to $\mathbf{R}$. The system of equations*

$$J(\Psi^* \hat{f}_{12}) = -[\gamma_t (f_{11} + \nu_x) - \gamma_x (f_{12} + \nu_t)], \tag{80}$$

$$(\Psi^* \hat{f}_{21})\gamma_x + (\Psi^* \hat{f}_{22})\delta_x = f_{21} \cosh \nu + f_{31} \sinh \nu, \tag{81}$$

$$(\Psi^* \hat{f}_{21})\gamma_t + (\Psi^* \hat{f}_{22})\delta_t = f_{22} \cosh \nu + f_{32} \sinh \nu, \tag{82}$$

$$(\Psi^* \hat{f}_{31})\gamma_x + (\Psi^* \hat{f}_{32})\delta_x = f_{21} \sinh \nu + f_{31} \cosh \nu, \tag{83}$$

$$(\Psi^* \hat{f}_{31})\gamma_t + (\Psi^* \hat{f}_{32})\delta_t = f_{22} \sinh \nu + f_{32} \cosh \nu, \tag{84}$$

in which the pull-backs of $\hat{u}$ and its derivatives with respect to $\hat{x}$, $\hat{t}$ appearing in the functions $(\Psi^* \hat{f}_{ij})(x, t)$ have been evaluated by means of the equation

$$\hat{u} \circ \Psi = \frac{1}{J}(\delta_t(f_{11} + \nu_x) - \delta_x(f_{12} + \nu_t)), \tag{85}$$

admits—whenever $u(x, t)$ is a I-generic solution of $\Xi = 0$—a local solution $\gamma(x, t)$, $\delta(x, t)$, $\nu(x, t)$ such that $\Psi(x, t) = (\gamma(x, t), \delta(x, t))$ is a local diffeomorphism.

Finally, the transformation theorem corresponding to these results is as follows:

Theorem 5. *Let $\Xi(x, t, u, \dots) = 0$ and $\hat{\Xi}(\hat{x}, \hat{t}, \hat{u}, \dots) = 0$ be two scalar equations describing Lorentzian pseudo-spherical surfaces of type I, and assume that $\Xi = 0$ has associated one-forms $\omega^i = f_{i1}\,dx + f_{i2}\,dt$, $i = 1, 2, 3$. Any I-generic solution $u(x, t)$ of $\Xi(x, t, u, \dots) = 0$, determines a I-generic solution $\hat{u}(\hat{x}, \hat{t})$ of $\hat{\Xi}(\hat{x}, \hat{t}, \hat{u}, \dots) = 0$ by*

$$\hat{u} \circ \Psi = \frac{1}{J}(\delta_t(f_{11} + \nu_x) - \delta_x(f_{12} + \nu_t)), \tag{86}$$

in which $\Psi(x, t) = (\gamma(x, t), \delta(x, t))$, and γ, δ, and ν are given by (80)–(84).

Example 5 (Constructing solitary waves). The linear equation

$$u_t = u_{xx} + u_x \tag{87}$$

describes pseudo-spherical surfaces with associated functions

$$f_{11} = u, \qquad f_{12} = u_x, \qquad f_{21} = -1, \qquad f_{22} = 0,$$
$$f_{31} = -u, \qquad f_{32} = -u_x. \tag{88}$$

The goal of this example is to determine a I-generic solitary wave solution of the elliptic sine-Gordon equation

$$\frac{\partial^2 \hat{u}}{\partial \hat{x}^2} + \frac{\partial^2 \hat{u}}{\partial \hat{t}^2} = \sin \hat{u} \, \cos \hat{u} \tag{89}$$

from a I-generic solution of the linear Eq. (87). To start with, note that Eq. (89) describes pseudo-spherical surfaces with associated functions

$$\hat{f}_{11} = \hat{u}_{\hat{t}}, \qquad \hat{f}_{12} = -\hat{u}_{\hat{x}}, \qquad \hat{f}_{21} = -\cos \hat{u}, \qquad \hat{f}_{22} = 0,$$
$$\hat{f}_{31} = 0, \qquad \hat{f}_{32} = -\sin \hat{u}. \tag{90}$$

Interestingly, the associated functions (90) *do not* satisfy the condition $\hat{f}_{11} = \hat{u}$ appearing in Lemma 4. Of course, one can arrange this by applying Lemma 2, but it is not obvious that the resulting system of Eqs. (80)–(84) can then be solved in closed form, to obtain an explicit solution of (89). A reasonable alternative is to suitably modify Lemma 4 and Theorem 5, and adapt them to the example at hand.

Lemma 4b. *Let $\Xi(x,t,u,\dots)=0$ and $\hat{\Xi}(\hat{x},\hat{t},\hat{u},\dots)=0$ be two scalar equations describing Lorentzian pseudo-spherical surfaces of type I with associated one-forms $\omega^i = f_{i1}\,dx + f_{i2}\,dt$ and $\hat{\omega}^i = \hat{f}_{i1}\,d\hat{x} + \hat{f}_{i2}\,d\hat{t}$, $i=1,2,3$, respectively, in which $\hat{f}_{21}=\hat{G}(\hat{u})$, $\hat{G}'\neq 0$. Let $\Psi(x,t)=(\gamma(x,t),\delta(x,t))$ be a smooth map with Jacobian $J=\gamma_x\delta_t-\gamma_t\delta_x$ from (an open subset of) M to (an open subset of) $\hat{M}$, and let ν be a smooth map from (an open subset of) M to $\mathbf{R}$. The system of equations*

$$(\Psi^*\hat{f}_{11})\gamma_x + (\Psi^*\hat{f}_{12})\delta_x = f_{11}+\nu_x, \tag{91}$$

$$(\Psi^*\hat{f}_{11})\gamma_t + (\Psi^*\hat{f}_{12})\delta_t = f_{12}+\nu_t, \tag{92}$$

$$-J(\Psi^*\hat{f}_{22}) = \gamma_t(f_{21}\cosh\nu + f_{31}\sinh\nu) - \gamma_x(f_{22}\cosh\nu + f_{32}\sinh\nu), \tag{93}$$

$$(\Psi^*\hat{f}_{31})\gamma_x + (\Psi^*\hat{f}_{32})\delta_x = f_{21}\sinh\nu + f_{31}\cosh\nu, \tag{94}$$

$$(\Psi^*\hat{f}_{31})\gamma_t + (\Psi^*\hat{f}_{32})\delta_t = f_{22}\sinh\nu + f_{32}\cosh\nu, \tag{95}$$

in which the pull-backs of $\hat{u}$ and its derivatives with respect to $\hat{x}$, $\hat{t}$ appearing in the functions $(\Psi^*\hat{f}_{ij})(x,t)$ have been evaluated by means of the equation

$$\hat{G}(\hat{u})\circ\Psi = \frac{1}{J}(\delta_t(f_{21}\cosh\nu + f_{31}\sinh\nu) - \delta_x(f_{22}\cosh\nu + f_{32}\sinh\nu)), \tag{96}$$

admits—whenever $u(x,t)$ is a I-generic solution of $\Xi=0$—a local solution $\gamma(x,t)$, $\delta(x,t)$, $\nu(x,t)$ such that $\Psi(x,t)=(\gamma(x,t),\delta(x,t))$ is a local diffeomorphism.

Theorem 5b. *Let $\Xi(x,t,u,\dots)=0$ and $\hat{\Xi}(\hat{x},\hat{t},\hat{u},\dots)=0$ be two scalar equations describing Lorentzian pseudo-spherical surfaces of type I, and assume that $\Xi=0$ has associated one-forms $\omega^i = f_{i1}\,dx + f_{i2}\,dt$, $i=1,2,3$. Any I-generic solution $u(x,t)$ of $\Xi(x,t,u,\dots)=0$, determines a I-generic solution $\hat{u}(\hat{x},\hat{t})$ of $\hat{\Xi}(\hat{x},\hat{t},\hat{u},\dots)=0$ by means of*

$$\hat{G}(\hat{u})\circ\Psi = \frac{1}{J}(\delta_t(f_{21}\cosh\nu + f_{31}\sinh\nu) - \delta_x(f_{22}\cosh\nu + f_{32}\sinh\nu)), \tag{97}$$

in which $\Psi(x,t)=(\gamma(x,t),\delta(x,t))$, and γ, δ, and ν are given by (91)–(95).

One can now check that if one considers the I-generic solution $u(x,t)=x+t$ of the linear Eq. (87), a solution to the system of equations appearing in Lemma 4b is the local diffeomorphism $\Psi(x,t)=(\gamma(x,t),\delta(x,t))$ given by

$$\hat{x} = \gamma(x,t) = \operatorname{arccosh}\left(\frac{e^{-x}}{(x+t)(-2+x+t)}\right), \tag{98}$$

$$\hat{t} = \delta(x,t) = \frac{e^{-x}(-1+x+t)}{(x+t)(-2+x+t)}, \qquad \nu = \ln|2/(x+t)-1|. \tag{99}$$

Formula (97) now yields

$$-\cos(\hat{u}\circ\Psi) = \mathrm{e}^{x}\sqrt{\mathrm{e}^{-2x}-(x+t)^2(-2+x+t)^2}, \tag{100}$$

and it follows, after inverting Eqs. (98) and (99), that $\hat{u}(\hat{x},\hat{t})$ is given by

$$\hat{u}(\hat{x},\hat{t}) = 2\,\arctan\,(\mathrm{e}^{\hat{x}}). \tag{101}$$

This is a solitary wave solution of the elliptic sine-Gordon equation.

Remark 6. The function $\hat{u}$ given by (101) is not a "traveling" wave. The reason for this is in the choice of associated functions (90): the speed v of the traveling wave solution of the elliptic sine-Gordon equation enters in the formulae for associated functions (90) as a parameter. The same parameter, in fact, which appears in the Bäcklund transformation for Eq. (89) (see [15]). For simplicity, it has been chosen $v = 0$.

4. A description of evolutionary PSS equations

Kamran and Tenenblat [13] performed a general classification of evolutionary PSS equations of the type $u_t = K(u, u_x, \dots, u_{x^k})$, assuming that their associated functions f_{ij} depend at most on $u, u_x, \dots, u_{x^k}$. They did not consider explicit x/t dependence neither in the associated one-forms nor in the equations themselves, however. Besides its geometrical importance, one can argue that this is a natural generalization because of two reasons: first, the analysis of second-order formally integrable equations carried out by Reyes [19] and Foursov et al. [10] shows that evolution equations with no explicit x/t dependence may well have x/t-dependent associated one-forms; second, there are interesting instances of x/t-dependent equations which are the integrability condition of a one-parameter family of $sl(2,\mathbf{R})$-valued linear problems, and which are not covered in the classification of PSS equations obtained so far. For example, the "spectral parameter" of the auxiliary linear problem may be a variable quantity, as in [2].

Evolutionary Riemannian PSS equations of the form $u_t = K(x, t, u, u_x, \dots, u_{x^k})$ are classified here, under the working assumption that the equation $u_t = K$ is encoded *exactly* in the structure equations

$$\mathrm{d}\omega^1 = \omega^3\wedge\omega^2, \qquad \mathrm{d}\omega^2 = \omega^1\wedge\omega^3, \qquad \mathrm{d}\omega^3 = \omega^1\wedge\omega^2, \tag{102}$$

in which the one-forms ω^i, $i = 1, 2, 3$, are considered as differential forms on a manifold J equipped with local coordinates $(x, t, u, u_x, u_{xx}, \dots, u_{x^k})$. One formalizes this idea thus [10,13].

Let $u_t = K(x, t, u, \dots, u_{x^k})$ be a kth order evolution equation, and consider the differential ideal $\mathcal{I}_K$ generated by the two-forms

$$\mathrm{d}u\wedge\mathrm{d}x + K(x, t, u, \dots, u_{x^k})\,\mathrm{d}x\wedge\mathrm{d}t,$$
$$\mathrm{d}u_{x^l}\wedge\mathrm{d}t - u_{x^{l+1}}\,\mathrm{d}x\wedge\mathrm{d}t, \quad 1\le l\le k-1,$$

on the reduced kth order jet space J with coordinates $x, t, u, u_x, \dots, u_{x^k}$, so that local solutions of $u_t = K$ correspond to integral submanifolds of the exterior differential system $\{\mathcal{I}_K, \mathrm{d}x \wedge \mathrm{d}t\}$. The ideal $\mathcal{I}_K$ will be called the *equation ideal* of $u_t = K$.

Definition 4. An evolution equation $u_t = K(x, t, u, \dots, u_k)$ is *strictly pseudo-spherical* if there exist one-forms $\omega^i = f_{i1}\,\mathrm{d}x + f_{i2}\,\mathrm{d}t$, $i = 1, 2, 3$, whose coefficients f_{ij} are smooth functions on the reduced kth order jet space J, such that the two-forms

$$\Omega_1 = \mathrm{d}\omega^1 - \omega^3 \wedge \omega^2, \quad \Omega_2 = \mathrm{d}\omega^2 - \omega^1 \wedge \omega^3, \quad \Omega_3 = \mathrm{d}\omega^3 - \omega^1 \wedge \omega^2, \tag{103}$$

generate the equation ideal $\mathcal{I}_K$.

It follows that the evolution equation $u_t = K$ is necessary and sufficient for the structure equations $\Omega_i = 0$, $i = 1, 2, 3$, to hold (so that the one-forms ω^i, $i = 1, 2, 3$, appearing in Definition 4 are target one-forms for $u_t = K$), that is, the equation ideal $\mathcal{I}_K$ is algebraically equivalent to a system of differential forms satisfying the pseudo-spherical structure equations if pulled back by local solutions of $u_t = K$. Definition 4 can be extended to other types of equations (see for instance [12,18]) but it seems that no general intrinsic characterization of strictly pseudo-spherical equations exists. In particular, it is not known whether there are evolutionary PSS equations that are not strictly pseudo-spherical.

The following lemma shows that the functions f_{ij} associated to strictly pseudo-spherical equations are strongly constrained.

Lemma 5. *Necessary and sufficient conditions for the equation $u_t = K(x, t, u, u_x, \dots, u_{x^k})$ to be strictly pseudo-spherical with associated functions f_{ij}, are the following:*

$$f_{i1,u_{x^l}} = f_{i2,u_{x^k}} = 0, \quad 1 \le l \le k-1, \quad i = 1, 2, 3, \tag{104}$$

$$f_{11,u}^2 + f_{21,u}^2 + f_{31,u}^2 \neq 0, \tag{105}$$

$$-f_{11,t} - f_{11,u}K + f_{12,x} + \sum_{l=0}^{k-1} f_{12,u_{x^l}} u_{x^{l+1}} - (f_{31}f_{22} - f_{32}f_{21}) = 0, \tag{106}$$

$$-f_{21,t} - f_{21,u}K + f_{22,x} + \sum_{l=0}^{k-1} f_{22,u_{x^l}} u_{x^{l+1}} - (f_{11}f_{32} - f_{12}f_{31}) = 0, \tag{107}$$

$$-f_{31,t} - f_{31,u}K + f_{32,x} + \sum_{l=0}^{k-1} f_{32,u_{x^l}} u_{x^{l+1}} - (f_{11}f_{22} - f_{12}f_{21}) = 0. \tag{108}$$

Moreover, if $u_t = K$ is strictly pseudo-spherical one can choose associated functions f_{ij} so that $f_{11,u} \neq 0$.

Proof. Suppose that $u_t = K(x, t, u, u_x, \dots, u_{x^k})$ is strictly pseudo-spherical with associated one-forms $\omega^i = f_{i1}\,\mathrm{d}x + f_{i2}\,\mathrm{d}t$. The structure equations $\Omega_i = 0$, in which the one-forms Ω_i are given by (103), imply the identities

$$-f_{11,t}\,\mathrm{d}x\wedge\mathrm{d}t+\sum_{l=0}^{k}f_{11,u_{x^l}}\,\mathrm{d}u_{x^l}\wedge\mathrm{d}x+f_{12,x}\,\mathrm{d}x\wedge\mathrm{d}t+\sum_{l=0}^{k}f_{12,u_{x^l}}\,\mathrm{d}u_{x^l}\wedge\mathrm{d}t-(f_{31}f_{22}-f_{32}f_{21})\,\mathrm{d}x\wedge\mathrm{d}t=0, \tag{109}$$

$$-f_{21,t}\,\mathrm{d}x\wedge\mathrm{d}t+\sum_{l=0}^{k}f_{21,u_{x^l}}\,\mathrm{d}u_{x^l}\wedge\mathrm{d}x+f_{22,x}\,\mathrm{d}x\wedge\mathrm{d}t+\sum_{l=0}^{k}f_{22,u_{x^l}}\,\mathrm{d}u_{x^l}\wedge\mathrm{d}t-(f_{11}f_{32}-f_{12}f_{11})\,\mathrm{d}x\wedge\mathrm{d}t=0, \tag{110}$$

$$-f_{31,t}\,\mathrm{d}x\wedge\mathrm{d}t+\sum_{l=0}^{k}f_{31,u_{x^l}}\,\mathrm{d}u_{x^l}\wedge\mathrm{d}x+f_{32,x}\,\mathrm{d}x\wedge\mathrm{d}t+\sum_{l=0}^{k}f_{32,u_{x^l}}\,\mathrm{d}u_{x^l}\wedge\mathrm{d}t-(f_{11}f_{22}-f_{12}f_{21})\,\mathrm{d}x\wedge\mathrm{d}t=0. \tag{111}$$

Substitution of the equations

$$\mathrm{d}u\wedge\mathrm{d}x=-K\,\mathrm{d}x\wedge\mathrm{d}t,\qquad \mathrm{d}u_{x^l}\wedge\mathrm{d}t=u_{x^{l+1}}\,\mathrm{d}x\wedge\mathrm{d}t,\quad 1\le l\le k-1$$

into (109)–(111), yields Eqs. (104), (106)–(108). On the other hand, at least one of the functions f_{i1}, $i=1,2,3$, must depend on the variable u, for, otherwise, it is a simple matter to convince oneself that (106)–(108) cannot be identities.

Conversely, to check that Eqs. (104)–(108) imply that $u_t=K$ is strictly pseudo-spherical with associated functions f_{ij}, is a straightforward computation.

Assume now that $u_t=K$ is strictly pseudo-spherical with associated one-forms $\omega^i=f_{i1}\,\mathrm{d}x+f_{i2}\,\mathrm{d}t$, $i=1,2,3$, so that the constraints (104)–(108) hold. There are two cases:

(a) If $f_{11,u}=0$ but $f_{21,u}\neq 0$, it is enough to use transformation (10) of Proposition 2 with $\rho=\pi/2$. One obtains new associated one-forms $\sigma^1=\omega^2$, $\sigma^2=-\omega^1$, and $\sigma^3=\omega^3$ satisfying the condition of the lemma.

(b) If $f_{11,u}=0$ and $f_{21,u}=0$, then $f_{31,u}\neq 0$ by (105). Use Proposition 2 again, this time apply transformation (11) with $\rho(x,t)$ determined by the relations

$$\rho_x+f_{21}=0,\qquad \rho_t+f_{22}\neq 0.$$

Obtain $\hat{\omega}^1=\omega^1\cosh\rho-\omega^3\sinh\rho$, $\hat{\omega}^2=(f_{22}+\rho_t)\,\mathrm{d}t$, and $\hat{\omega}^3=-\omega^1\sinh\rho+\omega^3\cosh\rho$. It is clear that $\hat{f}_{11,u}\neq 0$. It remains to check that $\hat{\omega}^1\wedge\hat{\omega}^2\not\equiv 0$, but this is obvious, since $\hat{\omega}^1\wedge\hat{\omega}^2=(f_{11}\cosh\rho-f_{31}\sinh\rho)(f_{22}+\rho_t)\,\mathrm{d}x\wedge\mathrm{d}t$, and this is not identically zero by construction. □

The following notation is used in formulating the classification theorems:

$$B_{i2}=\sum_{l=0}^{k-1}u_{x^{l+1}}f_{i2,u_{x^l}},\qquad E=\tfrac{1}{2}(f_{11}^2+f_{21}^2-f_{31}^2),\qquad H=\tfrac{1}{2}(f_{11}^2-f_{31}^2),$$
$$S=\tfrac{1}{2}(f_{11}^2+f_{21}^2),\qquad T=f_{21,u}f_{31,t}-f_{31,u}f_{21,t},\qquad Z=\tfrac{1}{2}(f_{11}^2+f_{21}^2+f_{31}^2).$$

The classification is divided in two branches, depending on whether $E_u=0$ or not. Because of Lemma 5, one can assume a priori that $f_{11,u}\neq 0$. No proofs are given, they are straightforward generalizations of the ones appearing in Kamran and Tenenblat's work [13].

1. $E_u\neq 0$. Define the functions J^l and C^l, $0\leq l\leq k-1$ recursively as follows:

$$J^{k-1}=\frac{1}{f_{11,u}}f_{31,u}f_{12,u},\qquad \text{and for}\ \ r\geq 1,$$
$$J^{r-1}=-\sum_{l=0}^{k-1}u_{x^{l+1}}J^r_{u_{x^l}}-J^r_x+\frac{1}{f_{11,u}}[f_{31,u}(B_{12}+f_{12,x})_{u_{x^r}}+H_uC^r$$
$$+f_{21}(f_{31,u}J^r-f_{11,u}f_{12,u_{x^r}})],$$

$$C^{k-1}=\frac{1}{f_{11,u}}f_{21,u}f_{12,u},\qquad \text{and for}\quad r\geq 1,$$
$$C^{r-1}=-\sum_{l=0}^{k-1}u_{x^{l+1}}C^r_{u_{x^l}}-C^r_x+\frac{1}{f_{11,u}}[f_{21,u}(B_{12}+f_{12,x})_{u_{x^r}}+S_uJ^r$$
$$+f_{31}(-f_{21,u}C^r-f_{11,u}f_{12,u_{x^r}})].$$

Theorem 6. *Let f_{ij}, $1\leq i\leq 3$, $1\leq j\leq 2$, be smooth functions of $x,t,u,\dots,u_{x^k}$ satisfying $f_{i1,u_{x^l}}=0$ for $1\leq l\leq k$, and $f_{i2,u_{x^k}}=0$. Assume, furthermore, that $f_{11,u}\neq 0$ and $E_u\neq 0$. The most general evolution equation $u_t=K(x,t,u,\dots,u_{x^k})$ describing pseudo-spherical surfaces with associated one-forms $\omega^i=f_{i1}\,\mathrm{d}x+f_{i2}\,\mathrm{d}t$ is given by*

$$u_t=\frac{1}{Z_u}[f_{11}(B_{12}+f_{12,x})+\sum_{l=0}^{k-1}u_{x^{l+1}}(f_{21}[C^l+f_{22,x}]+f_{31}[J^l+f_{32,x}])$$
$$+2f_{31}(-f_{11}f_{22}+f_{21}f_{12})-Z_t].\tag{112}$$

Moreover, the functions f_{22} and f_{32} must satisfy the constraints

$$f_{22}-\frac{1}{f_{11}E_u}(S_uf_{32,x}-f_{21}f_{31,u}f_{22,x})$$
$$=\frac{1}{f_{11}E_u}\left[-f_{11}f_{31,u}(B_{12}+f_{12,x})+S_u\left(\sum_{l=0}^{k-1}z_{l+1}J^i+f_{12}f_{21}\right)\right.$$
$$\left.-f_{21}f_{31,u}\left(\sum_{l=0}^{k-1}u_{x^{l+1}}C^l+f_{12}f_{31}\right)+S_tf_{31,u}-S_uf_{31,t}\right],$$

$$f_{32} - \frac{1}{f_{11}E_u}(f_{31}f_{21,u}f_{32,x} + H_u f_{22,x})$$
$$= \frac{1}{f_{11}E_u}\left[-f_{11}f_{21,u}(B_{12} + f_{12,x}) + f_{31}f_{21,u}\left(\sum_{l=0}^{k-1} u_{x^{l+1}}J^l + f_{12}f_{21}\right) + H_u\left(\sum_{l=0}^{k-1} u_{x^{l+1}}C^l + f_{12}f_{31}\right) + H_t f_{21,u} - H_u f_{21,t}\right],$$

and the functions f_{12} *and* f_{i1} *must satisfy the differential equations*

$$f_{32,z_r} = J^r, \qquad f_{22,z_r} = C^r, \quad \textit{for}\ \ 0 \le r \le k-1. \tag{113}$$

2. $E_u = 0$. Define the functions L^l, $0 \le l \le k-1$ recursively as follows:

$$L^{k-1} = \frac{1}{f_{11}}(f_{31}f_{32,u_{x^{k-1}}} - f_{21}f_{22,u_{x^{k-1}}}), \quad \text{and for}\ \ r \ge 1,$$
$$L^{r-1} = -\sum_{l=0}^{k-1} u_{x^{l+1}}L^r_{u_{x^l}} + L^r_x - \frac{1}{f_{11}}[f_{21}(B_{22} + f_{22,x})_{u_{x^r}} - f_{31}(B_{32} - f_{32,x})_{u_{x^r}}].$$

Theorem 7. *Let* f_{ij}, $1 \le i \le 3$, $1 \le j \le 2$, *be smooth functions of* $x, t, u, \dots, u_{x^k}$ *satisfying* $f_{i1,u_{x^l}} = 0$ *for* $1 \le l \le k$, *and* $f_{i2,u_{x^k}} = 0$. *Assume, furthermore, that* $f_{11,u} \ne 0$ *and* $E_u = 0$. *The most general evolution equation* $u_t = K(x, t, u, \dots, u_{x^k})$ *describing pseudo-spherical surfaces with associated one-forms* $\omega^i = f_{i1}\,\mathrm{d}x + f_{i2}\,\mathrm{d}t$ *is given by*

$$u_t = \frac{1}{f_{11,u}}\left[2\sum_{l=0}^{k-1} u_{x^{l+1}}L^l + f_{21}f_{32} - f_{22}f_{31} - f_{11,t} + \frac{1}{f_{11}}(f_{21}(B_{22} - f_{22,x}) - f_{31}(B_{32} - f_{32,x}) + E_t)\right]. \tag{114}$$

Moreover, the function f_{12} *is given by*

$$f_{12} = -\frac{1}{f_{11}f_{11,u}}[f_{31,u}(B_{22} + f_{22,x}) - f_{21,u}(B_{32} + f_{32,x}) - f_{11}(f_{31,u}f_{32} - f_{21,u}f_{22}) - T],$$

and the functions f_{22}, f_{32}, *and* f_{i1} *must satisfy the differential equations*

$$f_{12,u_{x^r}} = L^r, \quad \textit{for}\ \ 0 \le r \le k-1, \tag{115}$$

$$f_{12,x} = \sum_{l=0}^{k-1} u_{x^{l+1}}L^l + \frac{1}{f_{11}}(f_{21}(B_{22} - f_{22,x}) - f_{31}(B_{32} - f_{32,x}) + E_t). \tag{116}$$

This paper ends with two examples illustrating Theorems 6 and 7.

Example 6. The cylindrical KdV equation

$$\frac{\partial v}{\partial \sigma} = -\frac{\partial^3 v}{\partial \xi^3} - v\frac{\partial v}{\partial \xi} - \frac{1}{2\sigma}v$$

belongs to the first branch of the classification (Theorem 6). It describes pseudo-spherical surfaces with associated one-forms

$$\begin{aligned}\omega^1 &= -\frac{1}{4}\frac{2^{2/3}(-2+2^{5/3}\sigma v - 2^{2/3}\xi)}{\sqrt{\sigma}}\,\mathrm{d}\xi \\ &\quad + \frac{1}{36\sigma^{3/2}}(36\sigma^2\sqrt[3]{2}v_{\xi\xi} + 18\sqrt{\sigma} - 36\sigma^{3/2}v_\xi + 18(2^{2/3})\sigma v - 9\xi 2^{2/3} \\ &\quad + 12\sqrt[3]{2}\sigma^2 v^2 + 6\sqrt[3]{2}\sigma v\xi - 6\xi^2\sqrt[3]{2} + 2^{8/3}\sigma v - 11(2^{2/3})\xi - 28)\,\mathrm{d}\sigma, \\ \omega^2 &= \frac{1}{2}\frac{2^{2/3}}{\sqrt{\sigma}}\,\mathrm{d}\xi - \frac{1}{18\sigma^{3/2}}(3(2^{2/3})\sigma v + 3\xi 2^{2/3} + 3\sqrt{\sigma} - 6\sigma^{3/2}v_\xi + 14)\,\mathrm{d}\sigma,\end{aligned}$$

$$\begin{aligned}\omega^3 &= -\frac{2^{2/3}(-4+6(2^{2/3})\sigma v - 3(2^{2/3})\xi)}{12\sqrt{\sigma}}\,\mathrm{d}\xi \\ &\quad + \frac{1}{108\sigma^{3/2}}(108\sigma^2\sqrt[3]{2}v_{\xi\xi} + 54\sqrt{\sigma} - 108\sigma^{3/2}v_\xi + 54(2^{2/3})\sigma v - 27\xi 2^{2/3} \\ &\quad + 36\sqrt[3]{2}\sigma^2 v^2 + 18\sqrt[3]{2}\sigma v\xi - 18\xi^2\sqrt[3]{2} + 18(2^{2/3})\sigma v - 27(2^{2/3})\xi - 56)\,\mathrm{d}\sigma.\end{aligned}$$

Example 7. The evolution equation

$$v_t = \left(\frac{F - v_x}{x} + \frac{\partial F}{\partial x} + v_x\frac{\partial F}{\partial v} + v_{xx}\frac{\partial F}{\partial v_x}\right)v^2 - 3xv + x^2 v_x + cv,$$

in which F is an arbitrary smooth function depending at most on x, v, v_x, and c is a constant, belongs to the second branch of the classification (Theorem 7). Indeed (see [10]) it describes pseudo-spherical surfaces with associated one-forms

$$\omega^1 = -\frac{x\,\mathrm{e}^{-x}}{v}\,\mathrm{d}x - \frac{\mathrm{e}^{-x}(-xF(x,v,v_x)v + x^3 + v^2)}{v}\,\mathrm{d}t,$$

$$\omega^2 = \mathrm{d}x + c\,\mathrm{d}t, \qquad \omega^3 = \omega^1.$$

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References

[1] R. Beals, M. Rabelo, K. Tenenblat, Bäcklund transformations and inverse scattering solutions for some pseudospherical surface equations, Stud. Appl. Math. 81 (2) (1989) 125–151.

[2] S.P. Burtsev, V.E. Zakharov, A.V. Mikhailov, Inverse scattering method with variable spectral parameter, Teoret. i Mat. Fiz. 70 (3) (1987) 323–341.

[3] A.H. Chamseddine, D. Wyler, Topological gravity in $1 + 1$ dimensions, Nucl. Phys. B 340 (1990) 595–616.

[4] S.S. Chern, K. Tenenblat, Pseudo-spherical surfaces and evolution equations, Stud. Appl. Math. 74 (1986) 55–83.

[5] M. Crampin, Solitons and $SL(2, \mathbf{R})$, Phys. Lett. A 66 (1978) 170–172.

[6] M. Crampin, L. Hodgkin, D. Robinson, P. McCarthy, 2-manifolds of constant curvature, 3-parameter isometry groups and Bäcklund transformations, Rep. Math. Phys. 17 (3) (1980) 373–383.

[7] M. Crampin, F.A.E. Pirani, D.C. Robinson, The soliton connection, Lett. Math. Phys. 2 (1977) 15–19.

[8] L.P. Eisenhart, A Treatise on the Differential Geometry of Curves and Surfaces, Ginn and Company, Boston, 1909.

[9] L.D. Faddeev, L.A. Takhtajan, Hamiltonian Methods in the Theory of Solitons, Springer Series in Soviet Mathematics, Springer, Berlin, 1987.

[10] M.V. Foursov, P.J. Olver, E.G. Reyes, On formal integrability of evolution equations and local geometry of surfaces, Diff. Geom. Appl. 15 (2) (2001) 183–199.

[11] J. Gegenberg, G. Kunstatter, Solitons and black holes, Phys. Lett. B 413 (1997) 274–280.

[12] L. Jorge, K. Tenenblat, Linear problems associated to evolution equations of type $u_{tt} = F(u, u_x, u_{xx}, u_t)$, Stud. Appl. Math. 77 (1987) 103–117.

[13] N. Kamran, K. Tenenblat, On differential equations describing pseudospherical surfaces, J. Diff. Eq. 115 (1) (1995) 75–98.

[14] J. Krasil'shchik, A. Verbovetsky, Homological methods in equations of mathematical physics, Lectures given in August 1998 at the Diffiety Institute International Summer School, Levoča, Slovakia, Preprint DIPS 7/98, math.DG/9808130.

[15] G. Leibbrandt, Exact solutions of the elliptic sine equation in two space dimensions with application to the Josephson effect, Phys. Rev. B 15 (7) (1977) 3353–3361.

[16] P.J. Olver, Applications of Lie Groups to Differential Equations, 2nd Edition, Springer, Berlin, 1993.

[17] B. Palmer, Bäcklund transformations for surfaces in Minkowski space, J. Math. Phys. 31 (12) (1990) 2872–2875.

[18] M. Rabelo, On equations which describe pseudospherical surfaces, Stud. Appl. Math. 81 (1989) 221–248.

[19] E.G. Reyes, Pseudo-spherical surfaces and integrability of evolution equations, J. Diff. Eq. 147 (1) (1998) 195–230; Erratum: J. Diff. Eq. 153 (1) (1999) 223–224.

[20] E.G. Reyes, Conservation laws and Calapso–Guichard deformations of equations describing pseudo-spherical surfaces, J. Math. Phys. 41 (5) (2000) 2968–2989.

[21] E.G. Reyes, Some geometric aspects of integrability of differential equations in two independent variables, Acta Appl. Math. 64 (2–3) (2000) 75–109.

[22] E.G. Reyes, Integrability of evolution equations and pseudo-spherical surfaces, in: A. Coley, D. Levi, R. Milson, C. Rogers, P. Winternitz (Eds.), Centre de Recherches Mathématiques Proceedings and Lecture Notes, AMS, Providence, 2001.

[23] R. Sasaki, Soliton equations and pseudospherical surfaces, Nucl. Phys. B 154 (1979) 343–357.

[24] K. Tenenblat, Transformations of manifolds and applications to differential equations, Pitman Monographs and Surveys in Pure and Applied Mathematics, Vol. 93, 1998, Addison-Wesley/Longman, England.

[25] F. Warner, Foundations of Differentiable Manifolds and Lie Groups, Scott Foresman, Glenview, IL 1971.